Karlsruher Institut für Technologie

Hochperformante Humanoide Technologien

Robotics I: Introduction to Robotics
Exercise 3 – Inverse Kinematics und Dynamics

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http://www.humanoids.kit.edu

Task 1: Differential Inverse Kinematics

- SCARA-Robot with
  - One translational joint d_1
  - Two rotational joints θ_2, θ_3
  - Configuration $\boldsymbol{q} = (d_1,\ \theta_2,\ \theta_3)$

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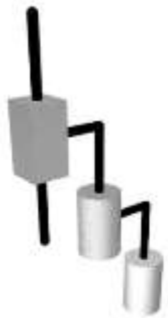

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- Forward Kinematics (position only):

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} -500 \cdot \sin(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \cos(\theta_2) \cdot \sin(\theta_3) - 500 \cdot \sin(\theta_2) \\ 500 \cdot \cos(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \sin(\theta_2) \cdot \sin(\theta_3) + 100 + 500 \cdot \cos(\theta_2) \\ d_1 \end{pmatrix}$$

End Effector Velocities

- The Jacobi matrix relates Cartesian end effector velocities to joint angular velocities

$$\dot{\boldsymbol{x}}(t) = J_f\big(\boldsymbol{\theta}(t)\big) \cdot \dot{\boldsymbol{\theta}}(t)$$

- The following problems can be solved with this relation:
  1. Given joint angular velocities, which kartesian end effector velocities are realized with them.
  2. Given a Cartesian end effector velocity, which joint angular velocities are required to realize them?

Task 1: Inverse Kinematics

- Forward Kinematics (position only):

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} -500 \cdot \sin(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \cos(\theta_2) \cdot \sin(\theta_3) - 500 \cdot \sin(\theta_2) \\ 500 \cdot \cos(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \sin(\theta_2) \cdot \sin(\theta_3) + 100 + 500 \cdot \cos(\theta_2) \\ d_1 \end{pmatrix}$$

- Approach for Inverse Kinematics

$$\dot{\boldsymbol{q}} = J^{-1}(\boldsymbol{q}) \cdot \dot{\boldsymbol{x}}$$

- Subtasks:
  - 1.1: Determine the inverse Jacobian matrix $J^{-1}(\boldsymbol{q})$.
  - 1.2: Determine $\dot{\boldsymbol{q}}$ for a given $\boldsymbol{q}$ and $\dot{\boldsymbol{x}}$.
  - 1.3: Which position exhibits singularities?

Task 1.1: Inverse Jacobian Matrix

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -500 \cdot \sin(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \cos(\theta_2) \cdot \sin(\theta_3) - 500 \cdot \sin(\theta_2) \\ 500 \cdot \cos(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \sin(\theta_2) \cdot \sin(\theta_3) + 100 + 500 \cdot \cos(\theta_2) \\ d_1 \end{pmatrix}$$

- Jacobian Matrix:

Task 1.1: Inverse Jacobian Matrix

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -500 \cdot \sin(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \cos(\theta_2) \cdot \sin(\theta_3) - 500 \cdot \sin(\theta_2) \\ 500 \cdot \cos(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \sin(\theta_2) \cdot \sin(\theta_3) + 100 + 500 \cdot \cos(\theta_2) \\ d_1 \end{pmatrix}$$

- Jacobian Matrix:

$$J(\boldsymbol{q}) = \left(\frac{\partial \boldsymbol{f}}{\partial d_1}, \frac{\partial \boldsymbol{f}}{\partial \theta_2}, \frac{\partial \boldsymbol{f}}{\partial \theta_3} \right)$$

$$\frac{\partial \boldsymbol{f}}{\partial d_1} =$$

Task 1.1: Inverse Jacobian Matrix

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- Jacobian Matrix:

$$J(\boldsymbol{q}) = \left(\frac{\partial \boldsymbol{f}}{\partial d_1}, \frac{\partial \boldsymbol{f}}{\partial \theta_2}, \frac{\partial \boldsymbol{f}}{\partial \theta_3} \right)$$

$$\frac{\partial \boldsymbol{f}}{\partial d_1} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial}{\partial d_1}(d_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Task 1.1: Inverse Jacobian Matrix $\frac{\partial f}{\partial \theta_2}$

- Simplify expression for x

$$x = -500 \cdot \sin(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \cos(\theta_2) \cdot \sin(\theta_3) - 500 \cdot \sin(\theta_2)$$

Task 1.1: Inverse Jacobian Matrix $\frac{\partial f}{\partial \theta_2}$

- Simplify expression for x

$$x = -500 \cdot \sin(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \cos(\theta_2) \cdot \sin(\theta_3) - 500 \cdot \sin(\theta_2)$$

$$= -500 \cdot (\sin(\theta_2) \cdot \cos(\theta_3) + \cos(\theta_2) \cdot \sin(\theta_3)) - 500 \cdot \sin(\theta_2)$$

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$$= -500 \cdot (\sin(\theta_2) \cdot \cos(\theta_3) + \cos(\theta_2) \cdot \sin(\theta_3)) - 500 \cdot \sin(\theta_2)$$

$$\boxed{\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta \\ \cos(\alpha + \beta) &= \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta \end{aligned}}$$

$$= -500 \cdot (\sin(\theta_2 + \theta_3)) - 500 \cdot \sin(\theta_2)$$

Task 1.1: Inverse Jacobian Matrix $\frac{\partial f}{\partial \theta_2}$

- Simplify expression for y

$$y = 500 \cdot \cos(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \sin(\theta_2) \cdot \sin(\theta_3) + 100 + 500 \cdot \cos(\theta_2)$$

Task 1.1: Inverse Jacobian Matrix $\frac{\partial f}{\partial \theta_2}$

- Simplify expression for y

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$$= 500 \cdot (\cos(\theta_2) \cdot \cos(\theta_3) - \sin(\theta_2) \cdot \sin(\theta_3)) + 100 + 500 \cdot \cos(\theta_2)$$

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- Simplify expression for y

$$y = 500 \cdot \cos(\theta_2) \cdot \cos(\theta_3) - 500 \cdot \sin(\theta_2) \cdot \sin(\theta_3) + 100 + 500 \cdot \cos(\theta_2)$$

$$= 500 \cdot (\cos(\theta_2) \cdot \cos(\theta_3) - \sin(\theta_2) \cdot \sin(\theta_3)) + 100 + 500 \cdot \cos(\theta_2)$$

$$\boxed{\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta \\ \cos(\alpha + \beta) &= \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta \end{aligned}}$$

$$= 500 \cdot \cos(\theta_2 + \theta_3) + 100 + 500 \cdot \cos(\theta_2)$$

Task 1.1: Inverse Jacobian Matrix $\frac{\partial f}{\partial \theta_2}$

$$x = -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2)$$

$$\frac{\partial x}{\partial \theta_2} =$$

$$y = 500 \cdot \cos(\theta_2 + \theta_3) + 100 + 500 \cdot \cos(\theta_2)$$

$$\frac{\partial y}{\partial \theta_2} =$$

$$z = d_1 \qquad \frac{\partial z}{\partial \theta_2} =$$

Task 1.1: Inverse Jacobian Matrix $\frac{\partial f}{\partial \theta_2}$

$$x = -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2)$$

$$\frac{\partial x}{\partial \theta_2} = -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2)$$

$$y = 500 \cdot \cos(\theta_2 + \theta_3) + 100 + 500 \cdot \cos(\theta_2)$$

$$\frac{\partial y}{\partial \theta_2} = -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2)$$

$$z = d_1 \qquad \frac{\partial z}{\partial \theta_2} = 0$$

Task 1.1: Inverse Jacobian Matrix $\frac{\partial f}{\partial \theta_3}$

$$x = -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2)$$

$$\frac{\partial x}{\partial \theta_3} =$$

$$y = 500 \cdot \cos(\theta_2 + \theta_3) + 100 + 500 \cdot \cos(\theta_2)$$

$$\frac{\partial y}{\partial \theta_3} =$$

$$z = d_1 \qquad \frac{\partial z}{\partial \theta_3} =$$

Task 1.1: Inverse Jacobian Matrix $\frac{\partial f}{\partial \theta_3}$

$$x = -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2)$$

$$\frac{\partial x}{\partial \theta_3} = -500 \cdot \cos(\theta_2 + \theta_3)$$

$$y = 500 \cdot \cos(\theta_2 + \theta_3) + 100 + 500 \cdot \cos(\theta_2)$$

$$\frac{\partial y}{\partial \theta_3} = -500 \cdot \sin(\theta_2 + \theta_3)$$

$$z = d_1 \qquad \frac{\partial z}{\partial \theta_3} = 0$$

Task 1.1: Inverse Jacobian Matrix

- Jacobian Matrix:

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

- Solve for:

$$J^{-1}(\boldsymbol{q}) = ?$$

Task 1.1: Inverse Jacobian Matrix

- Matrix:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

- Invert:

$$\mathrm{A}^{-1} = \frac{1}{\det A} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

- Determinant (Rule of Sarrus):

$$\det A = aei + bfg + cdh - ceg - bdi - afh$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = aei + bfg + cdh - ceg - bdi - afh$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = aei + bfg + cdh - ceg - bdi - afh$$

$$= 0 +$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = aei + bfg + cdh - ceg - bdi - afh$$

$$= 0 + (-500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3))$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = aei + bfg + cdh - ceg - bdi - afh$$

$$= 0 + (-500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3)) + 0$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = aei + bfg + cdh - ceg - bdi - afh$$

$$= 0 + (-500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3)) + 0$$

$$-(-500 \cdot \cos(\theta_2 + \theta_3)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2))$$

Task 1.1: Inverse Jacobian Matrix

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$$= 0 + (-500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3)) + 0$$

$$-(-500 \cdot \cos(\theta_2 + \theta_3)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2)) - 0$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = aei + bfg + cdh - ceg - bdi - afh$$

$$= 0 + (-500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3)) + 0$$

$$-(-500 \cdot \cos(\theta_2 + \theta_3)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2)) - 0 - 0$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \det J(\boldsymbol{q}) &= aei + bfg + cdh - ceg - bdi - afh \\ &= (-500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3)) \\ &\quad -(-500 \cdot \cos(\theta_2 + \theta_3)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2)) \end{aligned}$$

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$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

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$$= (-500) \cdot (-500) \cdot \begin{pmatrix} (\cos(\theta_2 + \theta_3) + \cos(\theta_2)) \cdot \sin(\theta_2 + \theta_3) \\ -(\sin(\theta_2 + \theta_3) + \sin(\theta_2)) \cdot \cos(\theta_2 + \theta_3) \end{pmatrix}$$

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$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

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$$= (-500) \cdot (-500) \cdot \begin{pmatrix} (\cos(\theta_2 + \theta_3) + \cos(\theta_2)) \cdot \sin(\theta_2 + \theta_3) \\ -(\sin(\theta_2 + \theta_3) + \sin(\theta_2)) \cdot \cos(\theta_2 + \theta_3) \end{pmatrix}$$

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$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

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$$= (-500) \cdot (-500) \cdot \begin{pmatrix} (\color{red}{\cos(\theta_2 + \theta_3)} + \cos(\theta_2)) \cdot \color{red}{\sin(\theta_2 + \theta_3)} \\ -(\color{blue}{\sin(\theta_2 + \theta_3)} + \sin(\theta_2)) \cdot \color{blue}{\cos(\theta_2 + \theta_3)} \end{pmatrix}$$

$$= 500^2 \cdot (\cos(\theta_2) \cdot \sin(\theta_2 + \theta_3) - \sin(\theta_2) \cdot \cos(\theta_2 + \theta_3))$$

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$$= 500^2 \cdot \begin{pmatrix} \cos(\theta_2) \cdot (\sin\theta_2 \cos\theta_3 + \cos\theta_2 \sin\theta_3) \\ - \sin(\theta_2) \cdot (\cos\theta_2 \cos\theta_3 - \sin\theta_2 \sin\theta_3) \end{pmatrix}$$

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$$\det J(\boldsymbol{q}) = 500^2 \cdot (\cos(\theta_2) \cdot \sin(\theta_2 + \theta_3) - \sin(\theta_2) \cdot \cos(\theta_2 + \theta_3))$$

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$$= 500^2 \cdot \begin{pmatrix} \cos(\theta_2) \cdot (\sin\theta_2 \cos\theta_3 + \cos\theta_2 \sin\theta_3) \\ - \sin(\theta_2) \cdot (\cos\theta_2 \cos\theta_3 - \sin\theta_2 \sin\theta_3) \end{pmatrix}$$

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$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = 500^2 \cdot (\cos(\theta_2) \cdot \sin(\theta_2 + \theta_3) - \sin(\theta_2) \cdot \cos(\theta_2 + \theta_3))$$

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$$= 500^2 \cdot \begin{pmatrix} \cos(\theta_2) \cdot (\sin\theta_2 \cos\theta_3 + \cos\theta_2 \sin\theta_3) \\ - \sin(\theta_2) \cdot (\cos\theta_2 \cos\theta_3 - \sin\theta_2 \sin\theta_3) \end{pmatrix}$$

$$= 500^2 \cdot (\cos^2\theta_2 \sin\theta_3 + \sin^2\theta_2 \sin\theta_3)$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = 500^2 \cdot (\cos(\theta_2) \cdot \sin(\theta_2 + \theta_3) - \sin(\theta_2) \cdot \cos(\theta_2 + \theta_3))$$

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$$= 500^2 \cdot \begin{pmatrix} \cos(\theta_2) \cdot (\sin\theta_2 \cos\theta_3 + \cos\theta_2 \sin\theta_3) \\ -\sin(\theta_2) \cdot (\cos\theta_2 \cos\theta_3 - \sin\theta_2 \sin\theta_3) \end{pmatrix}$$

$$= 500^2 \cdot (\cos^2\theta_2 \sin\theta_3 + \sin^2\theta_2 \sin\theta_3)$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = 500^2 \cdot (\cos(\theta_2) \cdot \sin(\theta_2 + \theta_3) - \sin(\theta_2) \cdot \cos(\theta_2 + \theta_3))$$

$$\boxed{\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta \\ \cos(\alpha + \beta) &= \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta \end{aligned}}$$

$$= 500^2 \cdot \begin{pmatrix} \cos(\theta_2) \cdot (\sin\theta_2 \cos\theta_3 + \cos\theta_2 \sin\theta_3) \\ - \sin(\theta_2) \cdot (\cos\theta_2 \cos\theta_3 - \sin\theta_2 \sin\theta_3) \end{pmatrix}$$

$$= 500^2 \cdot (\cos^2\theta_2 \sin\theta_3 + \sin^2\theta_2 \sin\theta_3)$$
$$= 500^2 \cdot \sin\theta_3 \, (\cos^2\theta_2 + \sin^2\theta_2)$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = 500^2 \cdot (\cos(\theta_2) \cdot \sin(\theta_2 + \theta_3) - \sin(\theta_2) \cdot \cos(\theta_2 + \theta_3))$$

$$\boxed{\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta \\ \cos(\alpha + \beta) &= \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta \end{aligned}}$$

$$= 500^2 \cdot \begin{pmatrix} \cos(\theta_2) \cdot (\sin\theta_2 \cos\theta_3 + \cos\theta_2 \sin\theta_3) \\ -\sin(\theta_2) \cdot (\cos\theta_2 \cos\theta_3 - \sin\theta_2 \sin\theta_3) \end{pmatrix}$$

$$= 500^2 \cdot (\cos^2\theta_2 \sin\theta_3 + \sin^2\theta_2 \sin\theta_3)$$
$$= 500^2 \cdot \sin\theta_3 (\cos^2\theta_2 + \sin^2\theta_2)$$
$$= 500^2 \cdot \sin\theta_3 \cdot 1$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = 500^2 \cdot \sin(\theta_3)$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{\det J(\boldsymbol{q})} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = 500^2 \cdot \sin(\theta_3)$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{\det J(\boldsymbol{q})} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

$$= \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

$$ei - fh = 0$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

$$ei - fh = 0$$
$$ch - bi = 0$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

$$ei - fh = 0$$
$$ch - bi = 0$$
$$cd - af = 0$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & bf - ce \\ fg - di & ai - cg & 0 \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

$$ei - fh = 0$$
$$ch - bi = 0$$
$$cd - af = 0$$
$$ae - bd = 0$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & bf - ce \\ fg - di & ai - cg & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$bf - ce =$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & bf - ce \\ fg - di & ai - cg & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$bf - ce = -500 \cdot (\cos(\theta_2 + \theta_3) + \cos(\theta_2)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3)) - (-500 \cdot \cos(\theta_2 + \theta_3)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2))$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500\cdot\cos(\theta_2+\theta_3) - 500\cdot\cos(\theta_2) & -500\cdot\cos(\theta_2+\theta_3) \\ 0 & -500\cdot\sin(\theta_2+\theta_3) - 500\cdot\sin(\theta_2) & -500\cdot\sin(\theta_2+\theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2\cdot\sin\theta_3}\begin{pmatrix} 0 & 0 & bf - ce \\ fg - di & ai - cg & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$bf - ce = -500\cdot(\cos(\theta_2+\theta_3)+\cos(\theta_2))\cdot(-500\cdot\sin(\theta_2+\theta_3)) - (-500\cdot\cos(\theta_2+\theta_3))\cdot(-500\cdot\sin(\theta_2+\theta_3) - 500\cdot\sin(\theta_2))$$

$$= 500^2\cdot(\cos(\theta_2+\theta_3)\cdot\sin(\theta_2+\theta_3) + \cos(\theta_2)\cdot\sin(\theta_2+\theta_3) - \cos(\theta_2+\theta_3)\cdot\sin(\theta_2+\theta_3) - \cos(\theta_2+\theta_3)\cdot\sin(\theta_2))$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500\cdot\cos(\theta_2+\theta_3) - 500\cdot\cos(\theta_2) & -500\cdot\cos(\theta_2+\theta_3) \\ 0 & -500\cdot\sin(\theta_2+\theta_3) - 500\cdot\sin(\theta_2) & -500\cdot\sin(\theta_2+\theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2\cdot\sin\theta_3}\begin{pmatrix} 0 & 0 & bf-ce \\ fg-di & ai-cg & 0 \\ dh-eg & bg-ah & 0 \end{pmatrix}$$

$$bf-ce = -500\cdot(\cos(\theta_2+\theta_3)+\cos(\theta_2))\cdot(-500\cdot\sin(\theta_2+\theta_3)) - (-500\cdot\cos(\theta_2+\theta_3))\cdot(-500\cdot\sin(\theta_2+\theta_3) - 500\cdot\sin(\theta_2))$$

$$= 500^2\cdot\begin{pmatrix} \cos(\theta_2+\theta_3)\cdot\sin(\theta_2+\theta_3) + \cos(\theta_2)\cdot\sin(\theta_2+\theta_3) \\ -\cos(\theta_2+\theta_3)\cdot\sin(\theta_2+\theta_3) - \cos(\theta_2+\theta_3)\cdot\sin(\theta_2) \end{pmatrix}$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & bf - ce \\ fg - di & ai - cg & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$bf - ce = -500 \cdot (\cos(\theta_2 + \theta_3) + \cos(\theta_2)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3)) - (-500 \cdot \cos(\theta_2 + \theta_3)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2))$$

$$= 500^2 \cdot \begin{pmatrix} \cos(\theta_2 + \theta_3) \cdot \sin(\theta_2 + \theta_3) + \cos(\theta_2) \cdot \sin(\theta_2 + \theta_3) \\ - \cos(\theta_2 + \theta_3) \cdot \sin(\theta_2 + \theta_3) - \cos(\theta_2 + \theta_3) \cdot \sin(\theta_2) \end{pmatrix}$$

$$= 500^2 \cdot (\cos(\theta_2) \cdot \sin(\theta_2 + \theta_3) - \cos(\theta_2 + \theta_3) \cdot \sin(\theta_2))$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ fg - di & ai - cg & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$bf - ce = -500 \cdot (\cos(\theta_2 + \theta_3) + \cos(\theta_2)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3)) - (-500 \cdot \cos(\theta_2 + \theta_3)) \cdot (-500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2))$$

$$= 500^2 \cdot \begin{pmatrix} \cos(\theta_2 + \theta_3) \cdot \sin(\theta_2 + \theta_3) + \cos(\theta_2) \cdot \sin(\theta_2 + \theta_3) \\ - \cos(\theta_2 + \theta_3) \cdot \sin(\theta_2 + \theta_3) - \cos(\theta_2 + \theta_3) \cdot \sin(\theta_2) \end{pmatrix}$$

$$= 500^2 \cdot (\cos(\theta_2) \cdot \sin(\theta_2 + \theta_3) - \cos(\theta_2 + \theta_3) \cdot \sin(\theta_2)) = 500^2 \cdot \sin\theta_3$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ fg - di & ai - cg & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$fg - di =$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ fg - di & ai - cg & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$fg - di = -500 \cdot \sin(\theta_2 + \theta_3) - 0$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ -500 \cdot \sin(\theta_2 + \theta_3) & ai - cg & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$ai - cg =$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ -500 \cdot \sin(\theta_2 + \theta_3) & ai - cg & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$ai - cg = 0 - (-500 \cdot \cos(\theta_2 + \theta_3) \cdot 1)$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ -500 \cdot \sin(\theta_2 + \theta_3) & ai - cg & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$ai - cg = 0 - (-500 \cdot \cos(\theta_2 + \theta_3) \cdot 1) = 500 \cdot \cos(\theta_2 + \theta_3)$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ -500 \cdot \sin(\theta_2 + \theta_3) & 500 \cdot \cos(\theta_2 + \theta_3) & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$dh - eg =$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ -500 \cdot \sin(\theta_2 + \theta_3) & 500 \cdot \cos(\theta_2 + \theta_3) & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$dh - eg = 0 \cdot 0 - (-500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2)) \cdot 1$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ -500 \cdot \sin(\theta_2 + \theta_3) & 500 \cdot \cos(\theta_2 + \theta_3) & 0 \\ dh - eg & bg - ah & 0 \end{pmatrix}$$

$$\begin{aligned} dh - eg &= 0 \cdot 0 - (-500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2)) \cdot 1 \\ &= 500 \cdot \sin(\theta_2 + \theta_3) + 500 \cdot \sin(\theta_2) \\ &= 500 \cdot (\sin(\theta_2 + \theta_3) + \sin(\theta_2)) \end{aligned}$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ -500 \cdot \sin(\theta_2 + \theta_3) & 500 \cdot \cos(\theta_2 + \theta_3) & 0 \\ 500 \cdot (\sin(\theta_2 + \theta_3) + \sin(\theta_2)) & bg - ah & 0 \end{pmatrix}$$

$$bg - ah =$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ -500 \cdot \sin(\theta_2 + \theta_3) & 500 \cdot \cos(\theta_2 + \theta_3) & 0 \\ 500 \cdot (\sin(\theta_2 + \theta_3) + \sin(\theta_2)) & bg - ah & 0 \end{pmatrix}$$

$$bg - ah = (-500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2)) \cdot 1 - 0 \cdot 0$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ -500 \cdot \sin(\theta_2 + \theta_3) & 500 \cdot \cos(\theta_2 + \theta_3) & 0 \\ 500 \cdot (\sin(\theta_2 + \theta_3) + \sin(\theta_2)) & bg - ah & 0 \end{pmatrix}$$

$$bg - ah = (-500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2)) \cdot 1 - 0 \cdot 0$$

$$= -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2)$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ -500 \cdot \sin(\theta_2 + \theta_3) & 500 \cdot \cos(\theta_2 + \theta_3) & 0 \\ 500 \cdot (\sin(\theta_2 + \theta_3) + \sin(\theta_2)) & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & 0 \end{pmatrix}$$

Task 1.1: Inverse Jacobian Matrix

$$J(\boldsymbol{q})^{-1} = \frac{1}{500^2 \cdot \sin\theta_3} \begin{pmatrix} 0 & 0 & 500^2 \cdot \sin\theta_3 \\ -500 \cdot \sin(\theta_2 + \theta_3) & 500 \cdot \cos(\theta_2 + \theta_3) & 0 \\ 500 \cdot (\sin(\theta_2 + \theta_3) + \sin(\theta_2)) & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & 0 \end{pmatrix}$$

$$J(\boldsymbol{q})^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -\dfrac{\sin(\theta_2 + \theta_3)}{500 \cdot \sin\theta_3} & \dfrac{\cos(\theta_2 + \theta_3)}{500 \cdot \sin\theta_3} & 0 \\ \dfrac{\sin(\theta_2 + \theta_3) + \sin(\theta_2)}{500 \cdot \sin\theta_3} & \dfrac{-\cos(\theta_2 + \theta_3) - \cos(\theta_2)}{500 \cdot \sin\theta_3} & 0 \end{pmatrix}$$

Task 1.2: Joint Angular Velocities

$$J(\boldsymbol{q})^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -\dfrac{\sin(\theta_2+\theta_3)}{500\cdot\sin\theta_3} & \dfrac{\cos(\theta_2+\theta_3)}{500\cdot\sin\theta_3} & 0 \\ \dfrac{\sin(\theta_2+\theta_3)+\sin(\theta_2)}{500\cdot\sin\theta_3} & \dfrac{-\cos(\theta_2+\theta_3)-\cos(\theta_2)}{500\cdot\sin\theta_3} & 0 \end{pmatrix}$$

- Given:
  - Robot state $\boldsymbol{q} = (d_1, \theta_2, \theta_3)^T = \left(1, 0, \frac{\pi}{2}\right)^T$
  - EEF Velocity $\dot{\boldsymbol{p}} = (1000, 0, 0)^T$
- Required:
  - Joint angular velocity $\dot{\boldsymbol{q}}$, which causes EEF velocity $\dot{\boldsymbol{p}}$

Task 1.2: Joint Angular Velocities

$$J(\boldsymbol{q})^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -\dfrac{\sin(\theta_2+\theta_3)}{500\cdot\sin\theta_3} & \dfrac{\cos(\theta_2+\theta_3)}{500\cdot\sin\theta_3} & 0 \\ \dfrac{\sin(\theta_2+\theta_3)+\sin(\theta_2)}{500\cdot\sin\theta_3} & \dfrac{-\cos(\theta_2+\theta_3)-\cos(\theta_2)}{500\cdot\sin\theta_3} & 0 \end{pmatrix}$$

$$J\left(\begin{pmatrix} 1 \\ 0 \\ \frac{\pi}{2} \end{pmatrix}\right)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -\dfrac{\sin(\theta_2+\theta_3)}{500\cdot\sin\theta_3} & \dfrac{\cos(\theta_2+\theta_3)}{500\cdot\sin\theta_3} & 0 \\ \dfrac{\sin(\theta_2+\theta_3)+\sin(\theta_2)}{500\cdot\sin\theta_3} & \dfrac{-\cos(\theta_2+\theta_3)-\cos(\theta_2)}{500\cdot\sin\theta_3} & 0 \end{pmatrix}$$

Task 1.2: Joint Angular Velocities

$$J\left(\begin{pmatrix} 1 \\ 0 \\ \frac{\pi}{2} \end{pmatrix}\right)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{\sin\left(0+\frac{\pi}{2}\right)}{500 \cdot \sin\frac{\pi}{2}} & \frac{\cos\left(0+\frac{\pi}{2}\right)}{500 \cdot \sin\frac{\pi}{2}} & 0 \\ \frac{\sin\left(0+\frac{\pi}{2}\right)+\sin(0)}{500 \cdot \sin\frac{\pi}{2}} & \frac{-\cos\left(0+\frac{\pi}{2}\right)-\cos(0)}{500 \cdot \sin\frac{\pi}{2}} & 0 \end{pmatrix}$$

Task 1.2: Joint Angular Velocities

$$J\left(\begin{pmatrix}1\\0\\\frac{\pi}{2}\end{pmatrix}\right)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -\dfrac{\sin\left(0+\frac{\pi}{2}\right)}{500\cdot\sin\frac{\pi}{2}} & \dfrac{\cos\left(0+\frac{\pi}{2}\right)}{500\cdot\sin\frac{\pi}{2}} & 0 \\ \dfrac{\sin\left(0+\frac{\pi}{2}\right)+\sin(0)}{500\cdot\sin\frac{\pi}{2}} & \dfrac{-\cos\left(0+\frac{\pi}{2}\right)-\cos(0)}{500\cdot\sin\frac{\pi}{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ -\dfrac{1}{500\cdot 1} & \dfrac{0}{500\cdot 1} & 0 \\ \dfrac{1+0}{500\cdot 1} & \dfrac{-0-1}{500\cdot 1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -\dfrac{1}{500} & 0 & 0 \\ \dfrac{1}{500} & -\dfrac{1}{500} & 0 \end{pmatrix}$$

Task 1.2: Joint Angular Velocities

$$\dot{\boldsymbol{q}} = J(\boldsymbol{q})^{-1} \cdot \dot{\boldsymbol{p}}$$

Task 1.2: Joint Angular Velocities

$$\dot{\boldsymbol{q}} = J(\boldsymbol{q})^{-1} \cdot \dot{\boldsymbol{p}}$$

$$= J\left(\begin{pmatrix} 1 \\ 0 \\ \frac{\pi}{2} \end{pmatrix}\right)^{-1} \cdot \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix}$$

Task 1.2: Joint Angular Velocities

$$\dot{\boldsymbol{q}} = J(\boldsymbol{q})^{-1} \cdot \dot{\boldsymbol{p}}$$

$$= J\left(\begin{pmatrix} 1 \\ 0 \\ \frac{\pi}{2} \end{pmatrix}\right)^{-1} \cdot \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{500} & 0 & 0 \\ \frac{1}{500} & -\frac{1}{500} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix}$$

Task 1.2: Joint Angular Velocities

$$\dot{\boldsymbol{q}} = J(\boldsymbol{q})^{-1} \cdot \dot{\boldsymbol{p}}$$

$$= J\left(\begin{pmatrix} 1 \\ 0 \\ \frac{\pi}{2} \end{pmatrix}\right)^{-1} \cdot \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{500} & 0 & 0 \\ \frac{1}{500} & -\frac{1}{500} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$$

Task 1.3: Singularities

- A kinematic chain is in a **singular configuration** if the corresponding Jacobian matrix is not of full rank.
  - Two or more columns of J_f are linear dependent
- The Jacobian is not invertible
  - Specific motions are not feasible
- In proximity of singularities **high joint angular velocities** can be required to maintain an end effector velocity

Task 1.3: Singularities

- A quadratic matrix $A \in \mathbb{R}^{n \times n}$ is of full rank if and only if its determinant is non-equal to zero.

$$\operatorname{rang} A = n \Leftrightarrow \det A \neq 0$$

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

Task 1.3: Singularities

- A quadratic matrix $A \in \mathbb{R}^{n \times n}$ is of full rank if and only if its determinant is non-equal to zero.

$$\operatorname{rang} A = n \Leftrightarrow \det A \neq 0$$

$$J(\boldsymbol{q}) = \begin{pmatrix} 0 & -500 \cdot \cos(\theta_2 + \theta_3) - 500 \cdot \cos(\theta_2) & -500 \cdot \cos(\theta_2 + \theta_3) \\ 0 & -500 \cdot \sin(\theta_2 + \theta_3) - 500 \cdot \sin(\theta_2) & -500 \cdot \sin(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det J(\boldsymbol{q}) = 500^2 \cdot \sin \theta_3$$

- For singularities $\boldsymbol{q}_{sing}$ of the quadratic matrix $J(\boldsymbol{q})$ we have:

$$\det J\left(\boldsymbol{q}_{sing}\right) = 500^2 \cdot \sin \theta_3 = 0$$

Task 1.3: Singularities

- For singularities $\boldsymbol{q}_{sing}$ of the quadratic matrix $J(\boldsymbol{q})$ we have:

$$\det J\left(\boldsymbol{q}_{sing}\right) = 500^2 \cdot \sin\theta_3 = 0$$

Task 1.3: Singularities

- For singularities $\boldsymbol{q}_{sing}$ of the quadratic matrix $J(\boldsymbol{q})$ we have:

$$\det J\left(\boldsymbol{q}_{sing}\right) = 500^2 \cdot \sin\theta_3 = 0$$

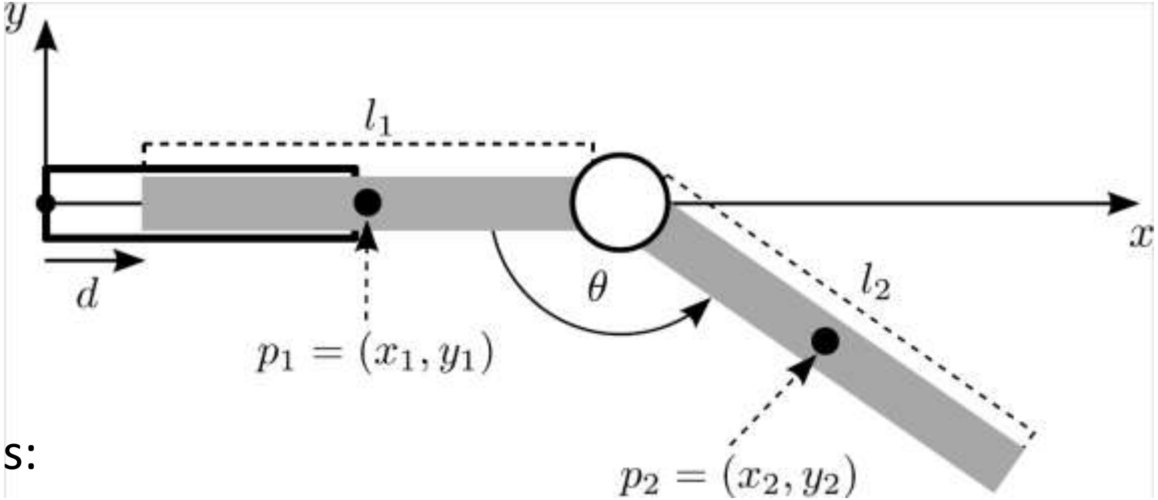
$$\sin\theta_3 = 0$$

$$\theta_3 = n \cdot \pi, n \in [0,1,2,\dots]$$

$$\theta_3 = 0 \vee \theta_3 = \pi, \qquad \theta_3 \in [0,2\pi)$$

$$\boldsymbol{q}_{sing,1} = \begin{pmatrix} d_1 \\ \theta_2 \\ 0 \end{pmatrix}, \boldsymbol{q}_{sing,2} = \begin{pmatrix} d_1 \\ \theta_2 \\ \pi \end{pmatrix}$$

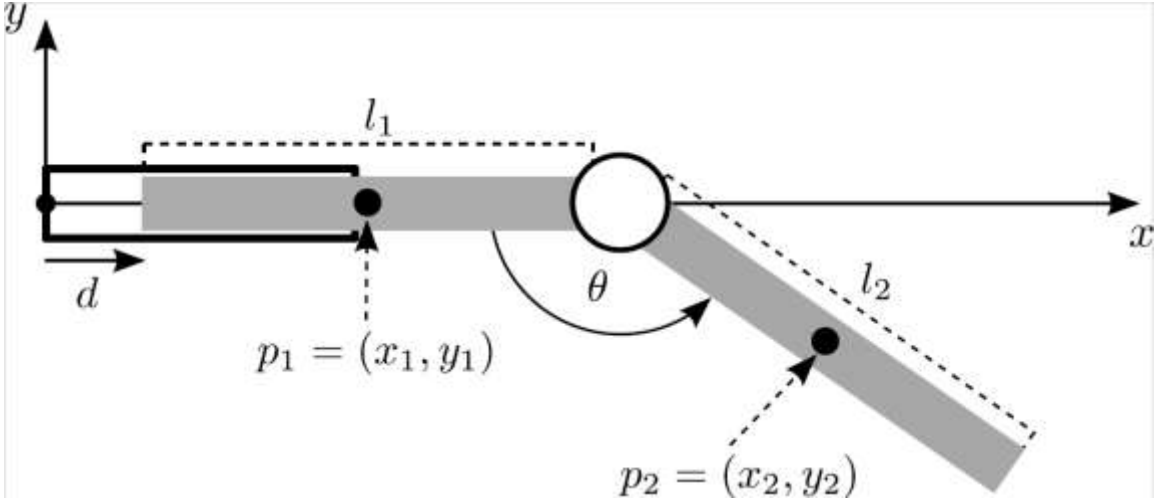
Task 2: Dynamic Modelling after Lagrange



- Assumptions:
  - Centers of mass in the middle of the segments
  - Negligible radius of the segments

- Configuration $\boldsymbol{q} = (d,\ \theta)^T$

Task 2: Dynamic Modelling after Lagrange



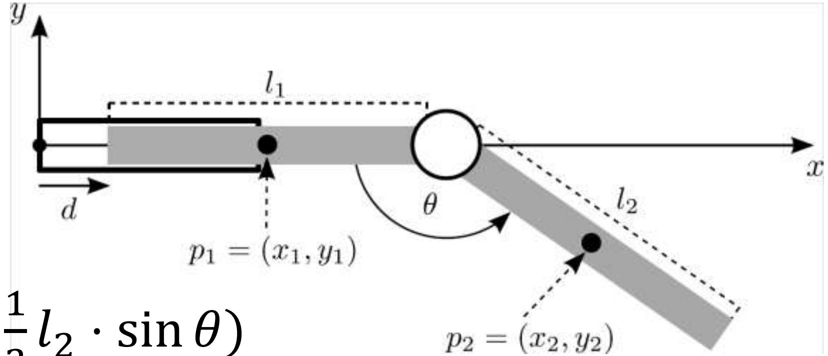
- Position of centers of mass:

$$p_1 = (x_1, y_1) = (\tfrac{1}{2} l_1 + d, 0)$$

$$p_2 = (x_2, y_2) = (l_1 + d - \tfrac{1}{2} l_2 \cdot \cos\theta\,, -\tfrac{1}{2} l_2 \cdot \sin\theta)$$

Task 2: Dynamic Modelling after Lagrange

- Configuration
  $\boldsymbol{q} = (d,\ \theta)^T$
- Position of centers of mass:
  $p_1 = (x_1, y_1) = (\frac{1}{2}l_1 + d, 0)$
  $p_2 = (x_2, y_2) = (l_1 + d - \frac{1}{2}l_2 \cdot \cos\theta\,, -\frac{1}{2}l_2 \cdot \sin\theta)$



- Model the dynamic of the robot system.
  - 2.1: Determine kinetic energy of each segment
  - 2.2: Determine potential energy for each segment
  - 2.3: Calculate Lagrange function
  - 2.4: Set up equation of motion

Method after Lagrange (Recap)

- Lagrange-Function:

$$\boxed{L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = E_{kin}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - E_{pot}(\boldsymbol{q})}$$

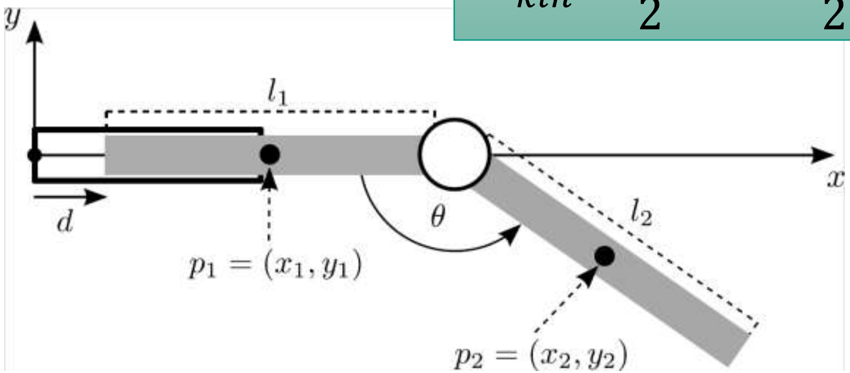
- Equation of motion:

$$\boxed{\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}}$$

- q_i: i-th component of the generalized coordinates
- τ_i: i-th component of the generalized forces

Task 2.1: Kinetic Energy

$$E_{kin} = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2$$



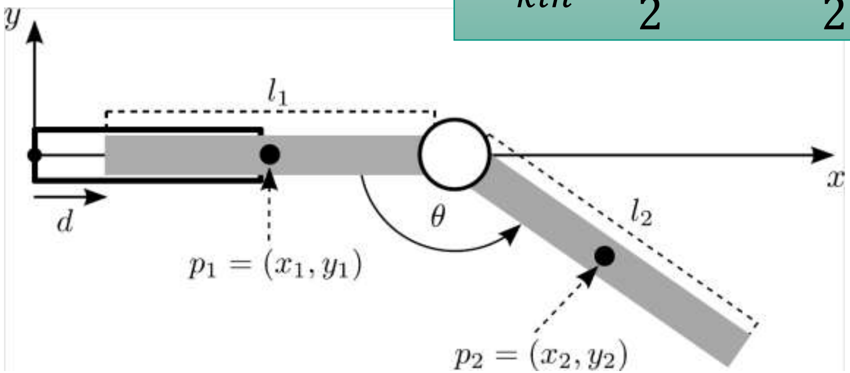
Kinetic energy for s_1 and s_2:

$E_{kin,1} =$

$E_{kin,2} =$

Task 2.1: Kinetic Energy

$$E_{kin} = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2$$



Kinetic energy for s_1 and s_2:

$$E_{kin,1} = \frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1 \dot{d}^2$$

$$E_{kin,2} = \frac{1}{2}m_2 v_2^2 + \frac{1}{2}J\omega_2^2 = \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}J\dot{\theta}^2$$

Task 2.1: Kinetic Energy

$$p_2 = (x_2, y_2) = \left(l_1 + d - \frac{1}{2} l_2 \cdot \cos\theta\,, -\frac{1}{2} l_2 \cdot \sin\theta\right)$$

$$\dot{x}_2 =$$

$$\dot{y}_2 =$$

Task 2.1: Kinetic Energy

$$p_2 = (x_2, y_2) = \left(l_1 + d - \frac{1}{2} l_2 \cdot \cos\theta \, , -\frac{1}{2} l_2 \cdot \sin\theta\right)$$

$$\dot{x}_2 = \dot{d} + \dot{\theta} \cdot \left(+\frac{1}{2} l_2 \sin\theta\right) = \dot{d} + \dot{\theta} \cdot \frac{1}{2} l_2 \sin\theta$$

$$\dot{y}_2 = \dot{\theta} \cdot \left(-\frac{1}{2} l_2 \cdot \cos\theta\right) = -\dot{\theta} \cdot \frac{1}{2} l_2 \cdot \cos\theta$$

Task 2.1: Kinetic Energy

$$\dot{x}_2 = \dot{d} + \dot{\theta} \cdot \frac{1}{2} l_2 \sin\theta$$

$$\dot{y}_2 = -\dot{\theta} \cdot \frac{1}{2} l_2 \cdot \cos\theta$$

$$E_{kin,2} = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} J \dot{\theta}^2 =$$

Task 2.1: Kinetic Energy

$$\dot{x}_2 = \dot{d} + \dot{\theta} \cdot \frac{1}{2} l_2 \sin\theta$$
$$\dot{y}_2 = -\dot{\theta} \cdot \frac{1}{2} l_2 \cdot \cos\theta$$

$$E_{kin,2} = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} J \dot{\theta}^2 =$$

$$= \frac{1}{2} m_2 \left(\left(\dot{d} + \dot{\theta} \cdot \frac{1}{2} l_2 \sin\theta \right)^2 + \left(-\dot{\theta} \cdot \frac{1}{2} l_2 \cdot \cos\theta \right)^2 \right) + \frac{1}{2} J \dot{\theta}^2$$

Task 2.1: Kinetic Energy

$$\dot{x}_2 = \dot{d} + \dot{\theta} \cdot \frac{1}{2} l_2 \sin\theta$$

$$\dot{y}_2 = -\dot{\theta} \cdot \frac{1}{2} l_2 \cdot \cos\theta$$

$$E_{kin,2} = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} J \dot{\theta}^2 =$$

$$= \frac{1}{2} m_2 \left(\left(\dot{d} + \dot{\theta} \cdot \frac{1}{2} l_2 \sin\theta \right)^2 + \left(-\dot{\theta} \cdot \frac{1}{2} l_2 \cdot \cos\theta \right)^2 \right) + \frac{1}{2} J \dot{\theta}^2$$

$$= \frac{1}{2} m_2 \left(\dot{d}^2 + 2 \cdot \dot{d} \cdot \dot{\theta} \cdot \frac{1}{2} l_2 \sin\theta + \dot{\theta}^2 \cdot \frac{1}{4} l_2^2 \sin^2\theta + \dot{\theta}^2 \cdot \frac{1}{4} l_2^2 \cdot \cos^2\theta \right) + \frac{1}{2} J \dot{\theta}^2$$

Task 2.1: Kinetic Energy

$$\dot{x}_2 = \dot{d} - \dot{\theta} \cdot \frac{1}{2} l_2 \sin\theta$$

$$\dot{y}_2 = -\dot{\theta} \cdot \frac{1}{2} l_2 \cdot \cos\theta$$

$$E_{kin,2} = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} J \dot{\theta}^2 =$$

$$= \frac{1}{2} m_2 \left(\left(\dot{d} + \dot{\theta} \cdot \frac{1}{2} l_2 \sin\theta \right)^2 + \left(-\dot{\theta} \cdot \frac{1}{2} l_2 \cdot \cos\theta \right)^2 \right) + \frac{1}{2} J \dot{\theta}^2$$

$$= \frac{1}{2} m_2 \left(\dot{d}^2 + 2 \cdot \dot{d} \cdot \dot{\theta} \cdot \frac{1}{2} l_2 \sin\theta + \dot{\theta}^2 \cdot \frac{1}{4} l_2^2 \sin^2\theta + \dot{\theta}^2 \cdot \frac{1}{4} l_2^2 \cdot \cos^2\theta \right) + \frac{1}{2} J \dot{\theta}^2$$

$$= \frac{1}{2} m_2 \dot{d}^2 + \frac{1}{2} m_2 \dot{d} \cdot \dot{\theta} l_2 \sin\theta + \frac{1}{2} m_2 \dot{\theta}^2 \cdot \frac{1}{4} l_2^2 + \frac{1}{2} J \dot{\theta}^2$$

Task 2.1: Kinetic Energy

$$E_{kin,2} = \frac{1}{2} m_2 \cdot \dot{d}^2 + \left(\frac{1}{2} m_2 l_2 \sin\theta\right) \cdot \dot{d}\dot{\theta} + \frac{1}{8} m_2 l_2^2 \cdot \dot{\theta}^2 + \frac{1}{2} \cdot \boldsymbol{J} \cdot \dot{\theta}^2$$

Task 2.1: Kinetic Energy

Moment of inertia for a rod with negligible radius, with respect to the center of gravity : $J = \frac{1}{12} m_2 l_2^2$

$$E_{kin,2} = \frac{1}{2} m_2 \cdot \dot{d}^2 + \left(\frac{1}{2} m_2 l_2 \sin\theta\right) \cdot \dot{d}\dot{\theta} + \frac{1}{8} m_2 l_2^2 \cdot \dot{\theta}^2 + \frac{1}{2} \cdot J \cdot \dot{\theta}^2$$

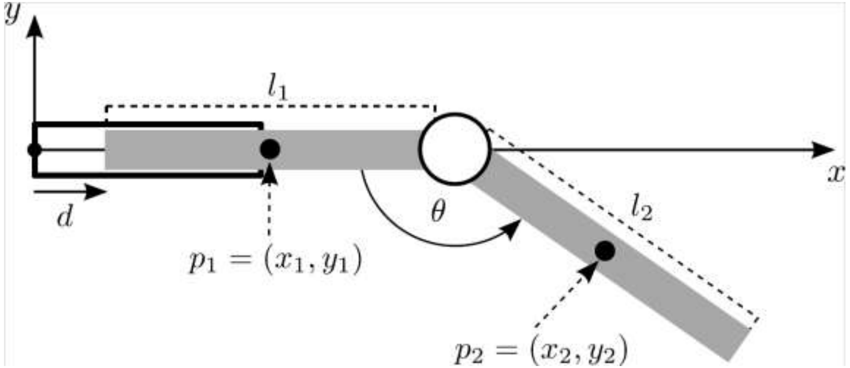
$$= \frac{1}{2} m_2 \cdot \dot{d}^2 + \left(\frac{1}{2} m_2 l_2 \sin\theta\right) \cdot \dot{d}\dot{\theta} + \frac{1}{8} m_2 l_2^2 \cdot \dot{\theta}^2 + \frac{1}{2} \cdot \frac{1}{12} m_2 l_2^2 \cdot \dot{\theta}^2$$

$$= \frac{1}{2} m_2 \cdot \dot{d}^2 + \left(\frac{1}{2} m_2 l_2 \sin\theta\right) \cdot \dot{d}\dot{\theta} + \frac{1}{8} m_2 l_2^2 \cdot \dot{\theta}^2 + \frac{1}{24} m_2 l_2^2 \cdot \dot{\theta}^2$$

$$= \frac{1}{2} m_2 \cdot \dot{d}^2 + \left(\frac{1}{2} m_2 l_2 \sin\theta\right) \cdot \dot{d}\dot{\theta} + \frac{1}{6} m_2 l_2^2 \cdot \dot{\theta}^2$$

Task 2.2: Potential Energy

$$E_{pot} = mgh$$



Potential Energies for s_1 und s_2:

$$E_{pot,1} = m_1 g y_1 = 0$$

$$E_{pot,2} = m_2 g y_2 = -\frac{1}{2} m_2 g l_2 \sin(\theta)$$

Task 2.3: Lagrange Function

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = E_{kin}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - E_{pot}(\boldsymbol{q})$$

$$E_{kin,1} = \frac{1}{2} m_1 \dot{d}^2$$

$$E_{pot,1} = 0$$

$$E_{kin,2} = \frac{1}{2} m_2 \cdot \dot{d}^2 + \left(\frac{1}{2} m_2 l_2 \sin\theta\right) \cdot \dot{d}\dot{\theta} + \frac{1}{6} m_2 l_2^2 \cdot \dot{\theta}^2$$

$$E_{pot,2} = -\frac{1}{2} m_2 g l_2 \sin(\theta)$$

$$L = E_{kin,1} + E_{kin,2} - E_{pot,1} - E_{pot,2} =$$

Task 2.3: Lagrange Function

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = E_{kin}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - E_{pot}(\boldsymbol{q})$$

$$E_{kin,1} = \frac{1}{2} m_1 \dot{d}^2 \qquad E_{kin,2} = \frac{1}{2} m_2 \cdot \dot{d}^2 + \left(\frac{1}{2} m_2 l_2 \sin\theta\right) \cdot \dot{d}\dot{\theta} + \frac{1}{6} m_2 l_2^2 \cdot \dot{\theta}^2$$

$$E_{pot,1} = 0 \qquad E_{pot,2} = -\frac{1}{2} m_2 g l_2 \sin(\theta)$$

$$L = \frac{1}{2} m_1 \dot{d}^2 + \frac{1}{2} m_2 \dot{d}^2 + \left(\frac{1}{2} m_2 l_2 \sin\theta\right) \cdot \dot{d}\dot{\theta} + \frac{1}{6} m_2 l_2^2 \cdot \dot{\theta}^2 + \frac{1}{2} m_2 g l_2 \sin(\theta)$$

$$= \frac{1}{2}(m_1 + m_2)\dot{d}^2 + \left(\frac{1}{2} m_2 l_2 \sin\theta\right) \cdot \dot{d}\dot{\theta} + \frac{1}{6} m_2 l_2^2 \cdot \dot{\theta}^2 + \frac{1}{2} m_2 g l_2 \sin(\theta)$$

Task 2.3: Differentiation of the Lagrange Function

$$L = \frac{1}{2}(m_1 + m_2)\dot{d}^2 + \frac{1}{2}m_2 l_2 \dot{d}\dot{\theta}\sin(\theta) + \frac{1}{6}m_2 l_2^2 \dot{\theta}^2 + \frac{1}{2}m_2 g l_2 \sin(\theta)$$

$$\frac{\partial L}{\partial \dot{d}} =$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{d}} =$$

$$\frac{\partial L}{\partial d} =$$

Task 2.3: Differentiation of the Lagrange Function

$$L = \frac{1}{2}(m_1 + m_2)\dot{d}^2 + \frac{1}{2}m_2 l_2 \dot{d}\dot{\theta}\sin(\theta) + \frac{1}{6}m_2 l_2^2 \dot{\theta}^2 + \frac{1}{2}m_2 g l_2 \sin(\theta)$$

$$\frac{\partial L}{\partial \dot{d}} = (m_1 + m_2)\dot{d} + \frac{1}{2}m_2 l_2\left(\dot{\theta} \cdot \sin(\theta)\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{d}} = (m_1 + m_2)\ddot{d} + \frac{1}{2}m_2 l_2\left(\ddot{\theta}\sin(\theta) + \dot{\theta} \cdot \cos(\theta) \cdot \dot{\theta}\right)$$

$$\frac{\partial L}{\partial d} = 0$$

Task 2.3: Differentiation of the Lagrange Function

$$L = \frac{1}{2}(m_1 + m_2)\dot{d}^2 + \frac{1}{2}m_2 l_2 \dot{d}\dot{\theta}\sin(\theta) + \frac{1}{6}m_2 l_2^2 \dot{\theta}^2 + \frac{1}{2}m_2 g l_2 \sin(\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} =$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} =$$

$$\frac{\partial L}{\partial \theta} =$$

Task 2.3: Differentiation of the Lagrange Function

$$L = \frac{1}{2}(m_1 + m_2)\dot{d}^2 + \frac{1}{2}m_2 l_2 \dot{d}\dot{\theta}\sin(\theta) + \frac{1}{6}m_2 l_2^2 \dot{\theta}^2 + \frac{1}{2}m_2 g l_2 \sin(\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m_2 l_2 \left(\dot{d} \cdot \sin(\theta)\right) + \frac{1}{3}m_2 l_2^2\, \dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m_2 l_2 \left(\ddot{d}\sin(\theta) + \dot{d}\dot{\theta}\cos(\theta)\right) + \frac{1}{3}m_2 l_2^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}m_2 l_2 \dot{d}\dot{\theta}\cos(\theta) + \frac{1}{2}m_2 l_2 g \cos(\theta)$$

Task 2.4: Equation of Motion

$$\boxed{\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}}$$

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{d}}\right) - \frac{\partial L}{\partial d}$$

Task 2.4: Equation of Motion

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{d}}\right) - \frac{\partial L}{\partial d}$$

$$= (m_1 + m_2)\ddot{d} + \frac{1}{2} m_2 l_2 \left(\ddot{\theta} \sin(\theta) + \dot{\theta}^2 \cos(\theta)\right) - 0$$

Task 2.4: Equation of Motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -\frac{1}{2}m_2 l_2\left(\ddot{d}\sin(\theta) + \dot{d}\dot{\theta}\cos(\theta)\right) + \frac{1}{3}m_2 l_2^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\frac{1}{2}m_2 l_2 \dot{d}\dot{\theta}\cos(\theta) + \frac{1}{2}m_2 l_2 g\cos(\theta)$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$

Task 2.4: Equation of Motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = -\frac{1}{2}m_2 l_2\left(\ddot d \sin(\theta) + \dot d\dot\theta\cos(\theta)\right) + \frac{1}{3}m_2 l_2^2 \ddot\theta$$

$$\frac{\partial L}{\partial \theta} = -\frac{1}{2}m_2 l_2 \dot d\dot\theta\cos(\theta) + \frac{1}{2}m_2 l_2 g\cos(\theta)$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta}$$

$$= \frac{1}{2}m_2 l_2\left(\ddot d \sin(\theta) + \dot d\dot\theta\cos(\theta)\right) + \frac{1}{3}m_2 l_2^2 \ddot\theta - \left(\frac{1}{2}m_2 l_2 \dot d\dot\theta\cos(\theta) + \frac{1}{2}m_2 l_2 g\cos(\theta)\right)$$

Task 2.4: Equation of Motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -\frac{1}{2}m_2 l_2\left(\ddot{d}\sin(\theta) + \dot{d}\dot{\theta}\cos(\theta)\right) + \frac{1}{3}m_2 l_2^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\frac{1}{2}m_2 l_2 \dot{d}\dot{\theta}\cos(\theta) + \frac{1}{2}m_2 l_2 g\cos(\theta)$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$

$$= \frac{1}{2}m_2 l_2 \ddot{d}\sin(\theta) + \frac{1}{2}m_2 l_2 \dot{d}\dot{\theta}\cos(\theta) + \frac{1}{3}m_2 l_2^2\ddot{\theta} - \left(\frac{1}{2}m_2 l_2 \dot{d}\dot{\theta}\cos(\theta) + \frac{1}{2}m_2 l_2 g\cos(\theta)\right)$$

$$= \frac{1}{2}m_2 l_2\sin(\theta)\,\ddot{d} + \frac{1}{3}m_2 l_2^2\ddot{\theta} - \frac{1}{2}m_2 l_2 g\cos(\theta)$$

Task 2.4: Equation of Motion

$$\boldsymbol{\tau} = M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\dot{\boldsymbol{q}}, \boldsymbol{q}) + \boldsymbol{g}(\boldsymbol{q})$$

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} (m_1 + m_2)\ddot{d} + \frac{1}{2} m_2 l_2 \left(\ddot{\theta}\sin(\theta) + \dot{\theta}^2 \cos(\theta)\right) \\ \frac{1}{2} m_2 l_2 \sin(\theta)\, \ddot{d} + \frac{1}{3} m_2 l_2^2 \ddot{\theta} - \frac{1}{2} m_2 l_2 g \cos(\theta) \end{pmatrix}$$

Task 2.4: Equation of Motion

$$\boldsymbol{\tau} = M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\dot{\boldsymbol{q}}, \boldsymbol{q}) + \boldsymbol{g}(\boldsymbol{q})$$

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} (m_1 + m_2)\ddot{d} + \frac{1}{2} m_2 l_2 \left(\ddot{\theta} \sin(\theta) + \dot{\theta}^2 \cos(\theta)\right) \\ \frac{1}{2} m_2 l_2 \sin(\theta)\, \ddot{d} + \frac{1}{3} m_2 l_2^2 \ddot{\theta} - \frac{1}{2} m_2 l_2 g \cos(\theta) \end{pmatrix}$$

Task 2.4: Equation of Motion

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\dot{\boldsymbol{q}}, \boldsymbol{q}) + \boldsymbol{g}(\boldsymbol{q})$$

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} (m_1 + m_2)\ddot{d} + \frac{1}{2} m_2 l_2 \sin(\theta)\,\ddot{\theta} + \frac{1}{2} m_2 l_2 \cos(\theta)\,\dot{\theta}^2 \\ \frac{1}{2} m_2 l_2 \sin(\theta)\,\ddot{d} + \frac{1}{3} m_2 l_2^2 \ddot{\theta} - \frac{1}{2} m_2 l_2 g \cos(\theta) \end{pmatrix}$$

This corresponds to the general Equation of Motion:

$$\boldsymbol{\tau} = \begin{pmatrix} m_1 + m_2 & \frac{1}{2} m_2 l_2 \sin(\theta) \\ \frac{1}{2} m_2 l_2 \sin(\theta) & \frac{1}{3} m_2 l_2^2 \end{pmatrix} \begin{pmatrix} \ddot{d} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} m_2 l_2 \cos(\theta)\,\dot{\theta}^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} m_2 l_2 \cos(\theta) \end{pmatrix}$$